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Using (Direct Computation, Variation Iteration, Successive Approximation and Regularization) Methods to Solve Linear Fredholm Integral Equation and Comparison of These Methods

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ABSTRACT

Integral equation is the equation in which the unknown function to be determined, appears under integral sign as it presented in introduction.

I discussed about linear Fredholm integral equation in which it is one kind of integral equation and solved this equation by different methods (Direct Computation, Variational Iteration, Successive approximation and the Regularization methods and comparison of these methods in order to solve linear Fredholm integral equation).

This paper has three parts:

First part: I introduced the Fredholm integral equation Second part is methods that is written above and on third part, I solved one example by these different methods and compare the methods.

Keywords- kernel, separable kernel, first and second kind, unknown function, Lagrange multiplier, variable, equivalent, converges, covert, effectively, correction function, same manner, approximation, real valued function, first approximation, Linear, consequent approximation.

I. INTRODUCTION

Integral equation is the equation in which the unknown function $u(x)$) to be determined, appears under integral sign, which has the form

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \qquad (1)$$

In classification of linear integral equation, Fredholm integral equation is one kind of it and it has the standard form that is given bellow.

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \quad a \le x \le b \quad (2)$$

Where the limitation of integral a, b ab, , are constant, $k(x,t)$ is the kernel of given integral equation, λ is a parameter, $f(x)$ is given advanced and the unknown function is linear.

If the unknown function be nonlinear, then the equation (2) is called nonlinear Fredholm integral equation and in this paper is discussed the linear Fredholm integral equation.

In equation (2) if $\phi(x) = 0$, then we have

$$f(x) + \lambda \int_a^b k(x,t)u(t)dt = 0 \qquad (3)$$

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The integral equation (3) is called Fredholm integral equation of the first kind, if $\phi(x) = 1$ then the integral (2) becomes

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad (4)$$

The integral equation (4) is called Fredholm integral equation of the second kind, in fact the (4) equation obtained from (2) by dividing both side of (2) by $\phi(x)$ provided that $\phi(x) \neq 0$.

If $f(x) = 0$ then the equation (4) is called homogenous Fredholm integral equation, if $f(x) \neq 0$ then the equation (4) is called non homogenous Fredholm integral equation.

In summary, the Fredholm integral equation is of the first kind if the unknown function $u(x)$ appears only under the integral sign. If the unknown function appears inside and outside of the integral sign, then the integral equation is called of the second kind. Rahman [2007]

II. METHODS OF SOLUTION

2.1 The direct computation method

In this part I introduced an efficient traditional method for solving the Fredholm integral equation, it is called the direct computation method, in this part my focused is on separable kernel $k(x,t)$ expressed as the form $k(x,t) = g(x)h(t)$, without losing the generality, what follows:

Consider: $u(x) = f(x) + \lambda g(x) \int_a^b h(t)u(t)dt \quad (1)$

The right hand side of integral (1) is depends on one variable t , this means that the definite integral in right hand side of (1) is equivalent to a numerical value of α , where α is constant , what follows

$$u(x) = f(x) + \lambda g(x) \int_a^b h(t)u(t)dt$$

Let: $\int_a^b h(t)u(t)dt = \alpha \qquad (2)$

$$\Rightarrow u(x) = f(x) + \lambda g(x)\alpha \qquad (3)$$

The solution of $u(x)$ completely determined form (3), upon evaluating the constant α , this is can be done by substituting equation (3) into equation (2) , I point out here that this approach is slightly different than other existing techniques, in this method I substitute (3) into (2) but not into (1) , it is worth nothing that the direct computation method determined the exact solution in a closed form.

2.2 Variational Iteration Method

The Variational iteration method gives successive approximation of the solution that may converges rapidly to the exact solution if such solution exists. However, for concrete problems the obtained approximation can be sued for numerical reason. For solving integral equation first we must covert the integral equation to its equivalent integro differential equation and then obtain the solution of Fredholm integral equation, this method works effectively if the kernel $k(x,t)$ is separable of the form $k(x,t) = g(x)h(t)$, consider the standard form of Fredholm integral equation bellow.

$$u(x) = f(x) + g(x) \int_a^b h(t)u(t)dt \quad , \quad k(x,t) = g(x)h(t) \qquad (1)$$

Differentiating both side of (1) with respect to x, we get

$$u'(x) = f'(x) + g'(x) \int_a^b h(t)u(t)dt \qquad (2)$$

The equation (1) is a Fredholm integro differential equation and its correction function is bellow

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \left\{ u_n'(t) - f'(t) - g'(t) \int_a^b h(r)u_n(r)dr \right\} dt \qquad (3)$$

Where λ is a general Lagrange multiplier. in Variation iteration method we follow two steps, first we determine the Lagrange multiplier, via Variational theory where integration by part should be taken and λ can be constant or function, in this paper for my present it is sufficient that $\lambda(t) = -1$ and there is computed some value of Lagrange multiplier bellow but it cannot cover all problems.

$$(1)\begin{cases} u' + f\big(u(t), u'(t)\big) = 0 \ , & \lambda = -1 \\ u_{n+1} = u_n(t) - \int_0^x [u_n' + f(u_n, u_n')]dt \end{cases}$$

$$(2)\begin{cases} u'' + f(u(t), u'(t), u''(t)) = 0 \ , & \lambda = (t-x) \\ u_{n+1} = u_n(t) - \int_0^x (t-x)[u_n'' + f(u_n, u_n', u_n'')]dt \end{cases}$$

$$(3)\begin{cases} u''' + f(u(t), u'(t), u''(t), u'''(t)) = 0 \ , & \lambda = -\frac{1}{2!}(t-x)^2 \\ u_{n+1} = u_n(t) - \int_0^x \frac{1}{2!}(t-x)^2[u_n'' + f(u_n, u_n', u_n'', u_n''')]dt \end{cases}$$

$$(4)\begin{cases} u^{(4)} + f\left(u(t), u'(t), u''(t), u'''(t), u_n^{(4)}(t)\right) = 0 \ , & \lambda = -\frac{1}{3!}(t-x)^3 \\ u_{n+1} = u_n(t) + \frac{1}{3!}\int_0^x \frac{1}{3!}(t-x)^3 \left[u_n''' + f\left(u_n, u_n', u_n'', u_n''', u_n^{(4)}\right)\right] dt \end{cases}$$

In generally we have

$$(5)\begin{cases} u^{(n)} + f\big(u(t), u'(t), u'''(t), \ldots, u_n^{(n)}\big) = 0 \ , & \lambda = (-1)^n \frac{1}{(n-1)!}(t-x)^{(n-1)} \\ u_{n+1} = u_n(t) + (-1)^n \int_0^x \frac{1}{(n-1)!}(t-x)^{(n-1)}[u_n''' + f\big(u_n, u_n', u_n'', \ldots, u_n^{(n)}\big)]dt \end{cases}$$

The correction function for $\lambda = -1$ is bellow

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{ u_n'(t) - f'(t) - g'(t) \int_a^b h(r)u_n(r)dr \right\} dt \qquad (4)$$

In (4) $u_{(0)}$ can be any selective function.

Second: equation (4) gives the successive approximation $u_n(x)$, by taking limit $\left(u(x) = \lim_{x\to\infty} u_n(x)\right)$ we can obtained the solution for equation (1), Abdul Majid Wazwaz,[2015]

2.3 The method of successive approximation

The successive approximation method, which was successfully applied to Volterra integral equation of the second kind, can be applied even more easily to the basic Fredholm integral equation of the second kind, consider the Fredholm integral equation bellow

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$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \quad (1)$$

I set the $f(x) = u_0(x)$ ($u_0(x)$ is called zero approximation and it can be any selected real value function, the first approximation $u_1(x)$ defined

$$u_1(x) = f(x) + \lambda \int_a^b k(x,t)u_0(t)dt \quad (2)$$

The second approximation $u_2(x)$ can be obtained by replacing $u_1(x)$ in equation (2) , what follow for the remaining approximation.

$$u_2(x) = f(x) + \lambda \int_a^b k(x,t)u_1(t)dt$$

$$u_3(x) = f(x) + \lambda \int_a^b k(x,t)u_2(t)dt \quad (3)$$

By continuing this process in the same manner, I can obtain the n^{th} approximation, what follows

$$u_n(x) = f(x) + \lambda \int_a^b k(x,t)u_{n-1}(t)dt \quad n \geq 1 \quad (4)$$

(4) is the common formula to obtain the approximation and the zero approximation is any real valued function, but most commonly selected function for $u_0(x)$ are $u_0(x) = 0,1,x$, if we chose $u_0(x) = 0$, then the first approximation $u_1(x) = f(x)$ and the solution of (1) obtain by the following formula, A.M.Wazwaz[2011]

$$u(x) = \lim_{n\to\infty} u_n(x)$$

2.4 The Regularization method

The regularization method was established independently by Tikhonov and Phillips, The regularization method, Consists of transforming first kind integral equations to second kind equations. The regularization method transforms the Linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b k(x,t)u(t)dt \quad (1)$$

to the approximation Fredholm integral equation

$$\alpha u_\alpha(x) = f(x) - \int_a^b k(x,t)u(t)dt \quad (2)$$

Where α is a small positive parameter called the regularization parameter. It is clear that (5) is an integral equation of the Second kind that can be rewritten as bellow

$$u_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{1}{\alpha} \int_a^b k(x,t)u_\alpha(t)dt \quad (3)$$

Moreover it was proved by Tikhonov and Phillips that the solution u_α of equation (3) converges to the solution $u(x)$ of (1) as $\alpha \to 0$, in other word

$$u(x) = \lim_{\alpha\to 0} u_\alpha(x)$$

It Important to note that the Fredholm integral equation of first is ill posed problem. The solution for an ill posed problem may not exist, and if it exist it may not be unique, as state before, we will apply the regularization method to transform the first kind of Fredholm integral equation to second kind and the second integral equation will be solved by the well-known existing method. A.N. Tiknovo [1963]

III. COMPARISIN OF METHODS BY EXAMPLE

Example: Use the regularization method to solve Fredholm integral equation of the first kind

$$2x = \int_0^1 xtu(t)dt \quad (1)$$

Solution: by the regularization method, we can write the equation (1) in the following form

$$\alpha u_\alpha(x) = 2x - \int_0^1 xtu_\alpha(t)dt \quad (2)$$

$$\Rightarrow u_\alpha(x) = \frac{2x}{\alpha} - \frac{1}{\alpha}\int_0^1 xtu_\alpha(t)dt \quad (3)$$

We first chose $u_0(x) = 0$, by putting this value of $u_0(x) = 0$ in equation (3), we can obtain the other consequent approximation, as it is the following

$$u_{\alpha_0}(x) = 0$$

$$u_{\alpha_1}(x) = \frac{2}{\alpha}x$$

$$u_{\alpha_2}(x) = \frac{2}{\alpha}x - \frac{2}{3\alpha^2}x + \frac{2}{9\alpha^3}x$$

$$u_{\alpha_4}(x) = \frac{2}{\alpha}x - \frac{2}{3\alpha^2}x + \frac{2}{9\alpha^3}x + \frac{2}{27\alpha^4}x$$

By following this manner we can obtain the approximation solution bellow

$$u_\alpha(x) = \frac{2}{\alpha}x\left(\frac{1}{(3\alpha)^0} - \frac{1}{3\alpha} + \frac{1}{(3\alpha)^2} - \frac{1}{(3\alpha)^3} + \cdots + (-1)^n \frac{1}{(3\alpha)^n}\right)$$

$$\Rightarrow u_\alpha(x) = \frac{2}{\alpha}x\left(\frac{3\alpha}{3\alpha+1}\right) \quad (4)$$

By taking limit from (4) as $\alpha \to 0$, we can obtain the exact solution

$$u(x) = \lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0}\frac{2}{\alpha}x\left(\frac{3\alpha}{3\alpha+1}\right) = 6x$$

$$\Rightarrow u(x) = 6x$$

Example: $u(x) = xe^x - x + x\int_0^1 u(t)dt$

By direct method:

$$u(x) = xe^x - x + x\int_0^1 u(t)dt \quad (1)$$

By suing direct method we can write the equation (1) as $u(x) = xe^x - x + x\alpha$ (2), where $\alpha = \int_0^1 u(t)dt$ (3). Putting the value of u x( ) from (2) into (3) , we can obtain the value of α and substitute back in (2) , what follows

$$\alpha = \int_0^1 (te^t - t + \alpha t)dt = te^t - e^t - \frac{1}{2}t^2 + \frac{\alpha}{2}t^2\Big|_0^1$$

$$\Rightarrow \alpha = \frac{\alpha}{2} + \frac{1}{2} \Rightarrow \alpha = 1 \quad (4)$$

By substituting the value of $\alpha = 1$ in (2) , we can obtain the exact solution bellow

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$$u(x) = xe^x - x + x\alpha \Rightarrow u(x) = xe^x - x(1) \Rightarrow u(x) = xe^x$$

By Variation Iteration Method:

$$u(x) = xe^x - x + x\int_0^1 u(t)dt \qquad (1)$$

By differentiating from both side of (1) with respect to x, we can obtain the Fredholm integro differential equation, what follows.

$$u'^{(x)} = xe^x + x - 1 + \int_0^1 u(t)dt \qquad (2)$$

Here $u_0(x) = 0$ and the correction function for (2) is bellow

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_n'(t) - te^t - e^t + 1 - \int_0^1 u_n(r)dr\right\}dt \qquad (3)$$

Here $\lambda = -1$

$$\Rightarrow u_1(x) = u_0(x) - \int_0^x \left\{u_0'(t) - te^t - e^t + 1 - \int_0^1 u_0(r)dr\right\}dt \Rightarrow u_1(x) = xe^x - x \qquad (4)$$

$$u_2(x) = u_1(x) - \int_0^x \left\{u_1'(t) - te^t - e^t + 1 - \int_0^1 u_1(r)dr\right\}dt \Rightarrow u_2(x) = xe^x - \frac{1}{2}x \qquad (5)$$

$$u_3(x) = u_2(x) - \int_0^x \left\{u_2'(t) - te^t - e^t + 1 - \int_0^1 u_2(r)dr\right\}dt \Rightarrow u_3(x) = xe^x - \frac{1}{4}x \qquad (6)$$

$$u_4(x) = u_3(x) - \int_0^x \left\{u_3'(t) - te^t - e^t + 1 - \int_0^1 u_3(r)dr\right\}dt \Rightarrow u_4(x) = xe^x - \frac{1}{8}x \qquad (7)$$

By (4),(5),(6),(7) we can write $u_n(x) = xe^x - \frac{1}{2^{n-1}}x \Rightarrow u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(xe^x - \frac{1}{2^{n-1}}\right) \Rightarrow u(x) = xe^x$

By Successive Approximation Method

$$u(x) = xe^x - x + x\int_0^1 u(t)dt \qquad (1)$$

For solving the (1) by successive approximation method, we have the formula

$$u_n(x) = f(x) + \lambda\int_a^b k(x,t)u_{n-1}(t)dt \qquad (2)$$

Here first approximation $(u_0(x))$ can be selected any real valued function but it is better to choose $u_0(x) = 0,1,x$. Consider $u_0(x) = 0$.

$$\Rightarrow u_1(x) = f(x) + \int_0^1 u_0(t)dt \Rightarrow u_1(x) = f(x) = xe^x - x \Rightarrow u_1(x) = xe^x - x \qquad (3)$$

$$u_2(x) = f(x) + \int_0^1 u_1(t)dt \Rightarrow u_2(x) = f(x) + \int_0^1 (te^t - t)dt \Rightarrow u_2(x) = xe^x - \frac{1}{2}x \qquad (4)$$

$$u_3(x) = f(x) + \int_0^1 u_2(t)dt \Rightarrow u_3(x) = f(x) + \int_0^1 \left(te^t - \frac{1}{2}t\right)dt \Rightarrow u_3(x) = xe^x - \frac{1}{2^2}x \qquad (5)$$

$$u_4(x) = f(x) + \int_0^1 u_3(t)dt \Rightarrow u_4(x) = f(x) + \int_0^1 \left(te^t - \frac{1}{2^2}t\right)dt \Rightarrow u_4(x) = xe^x - \frac{1}{2^3}x \qquad (6)$$

By continuing in this manner we can obtain the n^{th} approximation as bellow

$$u_n(x) = xe^x - \frac{1}{2^{n-1}}x \qquad (7)$$

$$\Rightarrow u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(xe^x - \frac{1}{2^{n-1}}x\right) \Rightarrow u(x) = xe^x$$

By regularization method, we can transfer the first kind of Fredholm integral equation to the second kind and after bringing changes we can use any better method, but the given example is the second kind and it is not need to use regularization method.

IV. CONCLUSION AND COMPARISON OF THES METHODS

A: without changing first kind of Fredholm integral equation to second kind (using regularization method) we can solve the equation .
B: in variation iteration method we should determine the Lagrange multiplier and it is different for every questions, we have to differentiate the given equation and change it to Fredholm integro differential equation and finally it need computing of limit ($u(x) = \lim_{x\to\infty}(u_n(x))$) .
This method need large value of computation.
C: the direct computation method should be used for Fredholm integral equation such that kernel of equation is separable.

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