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Preliminary Observations on the Novel Approaches in Fuzzy Conformable Differential Calculus

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ABSTRACT

This paper presents a novel extension of fuzzy calculus by integrating it with conformable calculus to introduce the fuzzy conformable derivative, a mathematical tool designed to handle the complexities of systems characterized by both uncertainty and fractional-order dynamics. The study begins by defining the fuzzy conformable derivative of order Ψ, which combines the ability of fuzzy calculus to model vagueness with the flexibility of conformable derivatives to capture non-integer order behaviors. This concept is further extended to higher orders, specifically the second order (2Ψ) and arbitrary order pΨ, enabling the modeling of more complex, multi-order dynamics in fuzzy-valued functions. Key contributions include the formal definitions of these derivatives, the establishment of their properties, and the development of operational rules that extend classical calculus operations to fuzzy systems. Additionally, the paper demonstrates how these derivatives can be applied to solve fuzzy differential equations and approximate functions using Taylor series expansions. While the fuzzy conformable derivative offers a powerful framework for modeling complex systems, the study also identifies certain limitations, such as potential unboundedness of solutions and the non-adherence to classical mathematical laws in higher-order cases. Overall, this work provides a comprehensive approach to differentiating fuzzy-valued functions in systems where uncertainty and fractional-order dynamics play a critical role. The proposed methods open up new avenues for research and application in fields such as control systems, economics, and engineering, where traditional calculus methods may not suffice. Future research directions include refining these methods, exploring computational techniques, and applying the framework to a broader range of real-world problems.

Keywords- Fuzzy Conformable Differential Calculus, Novel Approaches, Preliminary Observations, Fuzzy Logic Systems, Mathematical Analysis.

I. INTRODUCTION

In many real-world scenarios, the data we encounter is inherently imprecise or vague. For instance, consider the fluctuating water levels of a river or the varying temperature within a room—both phenomena defy exact measurement. Traditional mathematical tools, which rely on precise numbers, often struggle to model such uncertainty. This challenge is effectively addressed by fuzzy mathematics, a branch that utilizes fuzzy numbers to capture the essence of imprecise data. Fuzzy sets, initially introduced by Zadeh in the mid-20th century, extend the concept of classical sets to allow for degrees of membership, making them particularly useful for representing uncertain quantities[1]. Fuzzy calculus, a core area of fuzzy mathematics, builds upon these concepts to perform operations on fuzzy numbers. It was first developed by Chang and Zadeh, with further advancements by Bede and others who sought to overcome limitations such as the non-existence of derivatives for certain fuzzy-valued functions and the unbounded nature of solutions derived from these derivatives[2]. Despite these advancements, challenges remain in applying fuzzy calculus to solve differential equations, particularly due to the lack of a unified approach that combines the flexibility of fractional calculus with the specificity of fuzzy calculus. Fractional calculus, another extension of classical calculus, allows

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for the definition of derivatives and integrals of non-integer orders, offering a vast range of possibilities for modeling dynamic systems[3]. While fractional derivatives, such as those proposed by Riemann-Liouville and Caputo, have been instrumental in capturing complex phenomena across various fields, they are not without limitations. Issues such as the non-locality of these derivatives, and the complexities in applying standard mathematical rules (e.g., product rule, chain rule) hinder their broader application. Conformable calculus, a relatively recent development, addresses some of these limitations by defining fractional derivatives in a manner that closely resembles the classical derivative. This approach retains many of the desirable properties of traditional calculus, such as locality and ease of application, while extending the calculus to fractional orders[4]. The conformable derivative, first introduced by Khalil et al., has since attracted significant attention for its potential to model real-world problems with greater accuracy.

Given the complementary strengths of fuzzy calculus and conformable calculus, combining these two frameworks presents a promising avenue for addressing complex, real-world problems characterized by both uncertainty and fractional dynamics. The fuzzy conformable calculus merges the ability of fuzzy calculus to handle vagueness with the flexibility of conformable calculus to model non-integer order dynamics[5]. This paper builds on the initial attempts to integrate these fields by introducing the concept of a fuzzy conformable derivative of order 2Ψ and generalizing it to derivatives of arbitrary order pΨ. The primary contributions of this work include the establishment of fundamental properties of fuzzy conformable derivatives, the development of operational rules for these derivatives, and the application of these concepts to solve fuzzy conformable differential equations[6]. The structure of this paper is as follows: Section 2 provides an overview of the basic concepts in fuzzy and conformable differential calculus. Section 3 introduces the fuzzy conformable derivative of order Ψ, while Section 4 extends this to the second order (2Ψ). Section 5 generalizes the concept further to derivatives of order pΨ. Finally, the paper concludes with a summary of the key findings and potential future research directions.

II. BASIC CONCEPTS

This section lays the groundwork for the paper by introducing the foundational concepts of fuzzy calculus and conformable calculus, which are crucial for understanding the proposed fuzzy conformable calculus.

2.1 Fuzzy Calculus

Fuzzy calculus extends traditional calculus to handle data that is inherently uncertain or imprecise. This extension is built on the concept of fuzzy sets and fuzzy numbers[7], which generalize classical sets and real numbers to accommodate degrees of uncertainty.

Fuzzy Sets and Fuzzy Numbers

A fuzzy set, as initially defined by Zadeh, is a set where each element has a degree of membership represented by a value between 0 and 1, rather than a binary membership as in classical sets. Formally, a fuzzy set η is a mapping $\eta: \mathbb{R} \to [0, 1]$, where each real number is associated with a membership value that indicates the extent to which it belongs to the set.

A fuzzy number is a special type of fuzzy set that satisfies additional properties, making it suitable for representing imprecise quantities in a more structured way[8]. According to the definition provided by Dubois and Prade, a fuzzy number η must satisfy the following properties:

- **Convexity:** The membership function of a fuzzy number is convex, meaning that for any two points within the fuzzy number, the line segment connecting them lies within the fuzzy set.
- **Normality:** There exists at least one point in the fuzzy set where the membership value is 1.
- **Upper Semi-Continuity:** The membership function is upper semi-continuous, ensuring that small changes in the input do not lead to abrupt changes in the membership value.
- **Compact Support:** The fuzzy number is defined over a finite interval, beyond which the membership function is zero.

The γ-cut of a fuzzy number η, denoted as $[\eta]\gamma$, is a crucial concept in fuzzy calculus. It represents the set of all values that have a membership degree greater than or equal to γ. For example, the γ-cuts of a triangular fuzzy number η, defined by the ordered triple (a, b, c), are given by:

· $(\eta*)\gamma=a+(b-a)\gamma$
· $(\eta*)\gamma=c-(c-b)\gamma(\eta\hat{})\gamma = c - (c - b)\gamma(\eta*)\gamma=c-(c-b)\gamma$

These γ-cuts are used to perform arithmetic operations on fuzzy numbers, extending operations like addition, subtraction, and multiplication from real numbers to fuzzy numbers.

Arithmetic Operations on Fuzzy Numbers

The arithmetic operations on fuzzy numbers are generalized from operations on real intervals. For two fuzzy numbers η and υ, with γ-cuts $[\eta]\gamma = [(\eta_*)\gamma, (\eta^*)\gamma]$ and $[\upsilon]\gamma = [(\upsilon_*)\gamma, (\upsilon^*)\gamma]$, the following operations are defined:

- Addition: $[\eta + v]\ \gamma = [(\eta_*)\ \gamma + (v_*)\ \gamma, (\eta^*)\ \gamma + (v^*)\ \gamma]$
- Subtraction: $[\eta - v]\ \gamma = [(\eta_*)\ \gamma - (v_*)\ \gamma, (\eta^*)\ \gamma - (v^*)\ \gamma]$
- Scalar Multiplication: $[\alpha.\ \eta]\ \gamma = \begin{cases}[\alpha(\eta *)\gamma, \alpha(\eta *)\gamma], & if \alpha < 0 \\ [\alpha(\eta *)\gamma, \alpha(\eta *)\gamma], & if\ \alpha < 0\end{cases}$

These operations allow for the manipulation of fuzzy numbers in a manner analogous to real numbers but with the added capability of handling uncertainty.

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Fuzzy-Valued Functions and Derivatives

A fuzzy-valued function is a function that maps an interval of real numbers to the space of fuzzy numbers. Such functions can be represented in γ-cuts form as:

$$[\Phi(v)]\,\gamma = [(\Phi_*)\,\gamma(v), (\Phi^*)\,\gamma\,(v)]$$

The concept of differentiating fuzzy-valued functions is central to fuzzy calculus. The Hukuhara derivative (or H-derivative) is one of the earliest methods introduced to define the derivative of a fuzzy-valued function. However, this method is limited as it does not exist for all fuzzy-valued functions, and the solutions obtained are often unbounded.

To overcome these limitations, strongly generalized differentiability was introduced. A fuzzy-valued function Φ is strongly generalized differentiable at a point v_o if it satisfies certain conditions involving H-differences[9]. Depending on these conditions, the function can be classified as differentiable of type (1) or differentiable of type (2). This classification allows for a more comprehensive approach to differentiating fuzzy-valued functions, accommodating cases where traditional derivatives do not exist.

2.2 Conformable Calculus

Conformable calculus is a branch of fractional calculus that provides a natural extension of the classical derivative, enabling the definition of derivatives of non-integer orders while retaining many of the desirable properties of classical derivatives[10].

Conformable Derivative

The conformable derivative of order Ψ for a real-valued function Φ is defined as:

$\Phi\ \Psi(v) = \lim_{\theta\to 0} \frac{\Phi(v+\theta v^{1-\Psi})-\Phi(v)}{\theta}$, $\Psi \in (0,1)$.This definition ensures that the conformable derivative becomes identical to the classical derivative when Ψ = 1, thereby acting as a natural generalization.

Properties of Conformable Derivatives

- The conformable derivative maintains locality, making it suitable for investigating properties related to local scaling or fractional differentiability.
- It satisfies the relation: $\Phi\Psi(v) = v^{1-\Psi}\,\Phi'(v)$
- The conformable derivative also adheres to many classical calculus rules, such as the product rule and chain rule, which are often challenging to apply in other fractional derivatives[11].

Taylor Series in Conformable Calculus

Taylor's series can be extended to conformable calculus, where an infinitely differentiable function Φ can be expressed as: $\Phi(v) = \sum_{p=0}^{\infty} \frac{v^{p-p\Psi}(v-a)^p \Phi^p}{p!}(u)$. This series allows for the approximation of functions in a manner consistent with their fractional-order dynamics.

Example

For the exponential function in conformable calculus, the Maclaurin series expansion is given by:

$$e^{v\Psi}\ \Psi = \sum_{\prime=0}^{\infty} \frac{v^{(2'+1)\Psi}}{\prime!\Psi^{2'+1}}$$

This series illustrates how conformable derivatives can be used to approximate functions in fractional-order systems, providing a powerful tool for modeling dynamic phenomena.

III. FUZZY CONFORMABLE DERIVATIVE OF ORDER Ψ

The concept of the fuzzy conformable derivative integrates the ideas of fuzzy calculus and conformable calculus to address the complexities associated with modeling uncertain and fractional-order dynamics[12]. This section introduces the fuzzy conformable derivative of order Ψ, establishing its definition, properties, and significance in the context of fuzzy-valued functions.

3.1 Definition of Fuzzy Conformable Derivative of Order Ψ

The fuzzy conformable derivative of order Ψ generalizes the concept of differentiation to fuzzy-valued functions, incorporating the flexibility of conformable calculus while accommodating the inherent vagueness in fuzzy systems.

Definition 3.1: Fuzzy Conformable H-Differentiability

A fuzzy-valued function Φ is said to be conformable H-differentiable of order Ψ at a point v ∈ (a , b) if, for any $\theta > 0$, the H-differences $\Phi\,(v + \theta\, v^{1-\Psi})$ - $\Phi\,(v)$ and $\Phi\,(v)$ - $\Phi\,(v - \theta\, v^{1-\Psi})$ exist, and the following condition is satisfied: $\lim_{\theta\to 0} \frac{\Phi\,(v + \theta\, v^{1-\Psi}) - \Phi\,(v)}{\theta} = \lim_{\theta\to 0} \frac{\Phi\,(v) - \Phi\,(v - \theta\, v^{1-\Psi})}{\theta}$. This definition extends the classical notion of differentiability to fuzzy-valued functions, allowing for the analysis of such functions in systems where uncertainty is represented by fuzzy numbers.

3.2 Properties of Fuzzy Conformable Derivative

The fuzzy conformable derivative of order Ψ retains several key properties of both fuzzy calculus and conformable calculus, making it a powerful tool for modeling and analyzing fuzzy systems.

Relationship with Classical Derivatives

One of the important properties of the fuzzy conformable derivative is its consistency with classical derivatives when Ψ = 1. Specifically, for Ψ = 1, the fuzzy conformable derivative reduces to the classical fuzzy derivative, ensuring that the new definition acts as a natural extension rather than a replacement of existing concepts.

Property 3.2: Consistency with Classical Derivatives

$\Phi^{\Psi}(v) = v^{1-\Psi}\ \Phi'(v)$ This property highlights the close relationship between the conformable derivative and the classical derivative, demonstrating that the former can be viewed as a fractional-order generalization of the latter.

Taylor Series Expansion

The fuzzy conformable derivative can be used to extend the Taylor series expansion to fuzzy-valued functions,

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providing a method for approximating these functions in fractional-order systems.

Theorem 3.3: Taylor Series for Fuzzy-Valued Functions

The Taylor series for an infinitely differentiable fuzzy-valued function Φ with respect to the conformable derivative at a point Φ(v) is given by: $\Phi(v) = \sum_{p=0}^{\infty} \frac{v^{p-p\Psi}(v-a)^p \Phi^p}{p!}(u)$. This expansion allows for the representation of fuzzy-valued functions in a series form, facilitating their use in mathematical modeling and analysis.

3.3 Examples of Fuzzy Conformable Derivatives

To illustrate the application of the fuzzy conformable derivative, consider the following example.

Example 3.4: Fuzzy Conformable Derivative of a Simple Function

Let Φ (v) = η $\times\varphi(v)$, where η is a fuzzy number, and φ (v) is a real-valued function that is Ψ-differentiable. The fuzzy conformable derivative of Φ (v) is given by: Φ^{Ψ} (v) = η $\times\varphi^{\Psi}$(v) This example demonstrates how the fuzzy conformable derivative can be applied to a product of a fuzzy number and a real-valued function, extending the differentiation process to fuzzy systems.

3.4 Limitations and Generalizations

While the fuzzy conformable derivative provides a versatile tool for analyzing fuzzy systems, it also has certain limitations. For example, the derivative may not always produce bounded solutions, and certain classical properties, such as the index law or commutative law, may not hold in the fuzzy conformable context.

Remark 3.5: Limitations of Fuzzy Conformable Derivatives

- **Bounded Solutions:** The fuzzy conformable derivative may yield solutions that are unbounded, particularly in cases where the underlying fuzzy-valued function has specific characteristics that defy conventional bounds.
- **Failure of Classical Laws:** The fuzzy conformable derivative does not necessarily obey the index law or the commutative law, highlighting the need for careful consideration when applying these derivatives in mathematical models.

To address these limitations, the concept of strongly generalized Ψ-differentiability is introduced. This generalization provides a broader framework for differentiating fuzzy-valued functions[13], allowing for the application of fuzzy conformable derivatives in cases where the standard approach may not suffice.

Definition 3.6: Strongly Generalized Ψ-Differentiability

A fuzzy-valued function Φ is strongly generalized Ψ-differentiable if it satisfies a set of conditions that extend beyond the standard definition of the fuzzy conformable derivative. These conditions allow for the differentiation of functions that do not meet the criteria for H-differentiability, thereby expanding the applicability of fuzzy conformable calculus.

IV. FUZZY CONFORMABLE DERIVATIVE OF ORDER 2Ψ

Building upon the concept of the fuzzy conformable derivative of order Ψ, this section extends the framework to the second-order derivative, denoted as 2Ψ[14]. This extension allows for more complex modeling of fuzzy-valued functions, particularly in systems where the dynamics are governed by higher-order processes.

4.1 Definition of Fuzzy Conformable Derivative of Order 2Ψ

The fuzzy conformable derivative of order 2Ψ generalizes the concept of differentiation to capture the second-order dynamics of fuzzy-valued functions within the context of conformable calculus.

Definition 4.1: Strongly Generalized Conformable Derivative of Order 2Ψ

A fuzzy-valued function Φ is said to be strongly generalized differentiable of order 2Ψ if there exists a fuzzy number Φ 2Ψ(v) such that the following conditions hold:

1. For any $\theta > 0$, the H-differences $\Phi^{\Psi}(v + \theta v^{1-\Psi}) - \Phi^{\Psi}(v)$ and $\Phi^{\Psi}(v) - \Phi^{\Psi}(v - \theta v^{1-\Psi})$ exist, and the limit is equal to $\Phi^{2\Psi}$ (v).

$$\lim_{\theta\to 0} \frac{\Phi^{\Psi}(v + \theta v^{1-\Psi}) - \Phi^{\Psi}(v)}{\theta} = \lim_{\theta\to 0} \frac{\Phi^{\Psi}(v) - \Phi^{\Psi}(v - \theta v^{1-\Psi})}{\theta}$$

2. If Φ (v) and $\Phi^{\Psi}(v)$are both differentiable of the same type, then the second-order derivative can be written as:

$$\lim_{\theta\to 0} \frac{\Phi(v + 2\theta v^{1-\Psi}) + \Phi(v) - 2\Phi(v + \theta v^{1-\Psi})}{-\theta^2} = \Phi^{2\Psi}(v)$$

3. If Φ (v) is differentiable of one type and $\Phi^{\Psi}(v)$ is differentiable of another, then the second-order derivative can be expressed as:

$$\lim_{\theta\to 0} \frac{2\Phi(v + \theta v^{1-\Psi}) - \Phi(v) - \Phi(v + 2\theta v^{1-\Psi})}{-\theta^2} = \Phi^{2\Psi}(v)$$

This definition expands the conformable derivative to account for second-order effects, which are crucial in modeling more complex fuzzy systems.

4.2 Properties of the Fuzzy Conformable Derivative of Order 2Ψ

The fuzzy conformable derivative of order 2Ψ retains several key properties that are analogous to those of the first-order derivative but extends them to accommodate second-order dynamics.

Property 4.2: Relation Between First-Order and Second-Order Derivatives

For a fuzzy-valued function Φ that is strongly generalized conformable differentiable of order Ψ, the second-order derivative can be related to the first-order derivative as follows:

1. If $\Phi^{\Psi}(v)$is differentiable of type (Ψ-1), then:

$\Phi_1^{2\Psi}(v) = v^{2(1-\Psi)} \Phi_1^2(v)$

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Where Φ_1^2 (v) denotes the second classical derivative of Φ with respect to v.

2. If $\Phi^{\Psi}(v)$ is differentiable of type (Ψ-2), then:

$\Phi_2^{2\Psi}$ (v) $=v^{2(1-\Psi)}\,\Phi_2^2$ (v)

where Φ_2^2 (v) is the second classical derivative of Φ with respect to v, but in the type (Ψ-2) context.

These relations provide a bridge between first-order and second-order fuzzy conformable derivatives, allowing for the systematic extension of differentiation to higher orders.

Taylor Series Expansion for Second-Order Derivatives

The second-order fuzzy conformable derivative can be used to extend the Taylor series expansion to account for second-order effects[15]. For an infinitely differentiable fuzzy-valued function Φ (v) , the Taylor series can be expressed as:

$$\Phi(\mathrm{v}) = \sum_{p=0}^{\infty} \frac{v^{p-p\Psi}(v-u)^p \Phi^{p2\Psi}}{p!}(u)$$

This expansion facilitates the approximation of fuzzy-valued functions in systems where second-order dynamics are significant.

4.3 Examples and Applications

To demonstrate the application of the fuzzy conformable derivative of order 2Ψ, consider the following example.

Example 4.3: Second-Order Derivative of a Fuzzy-Valued Function

Let Φ (v) = η ×ψ(v), where η is a fuzzy number, and ψ(v) is a real-valued function that is differentiable of order 2Ψ. The second-order fuzzy conformable derivative of Φ (v) is given by:$\Phi^{2\Psi\,(v)} = \eta \times \psi^{2\Psi}(v)$

This example illustrates how the second-order fuzzy conformable derivative can be applied to a product of a fuzzy number and a real-valued function, extending the analysis to second-order effects.

4.4 Limitations and Extensions

While the fuzzy conformable derivative of order 2Ψ provides a powerful tool for modeling second-order dynamics in fuzzy systems, it is important to acknowledge its limitations. Similar to the first-order derivative, the second-order derivative may not always produce bounded solutions, and certain classical properties may not hold in this context.

Remark 4.4: Limitations and Extensions of the Second-Order Fuzzy Conformable Derivative

- Bounded Solutions: The second-order fuzzy conformable derivative may yield unbounded solutions in certain cases, particularly when the underlying fuzzy-valued function exhibits extreme behaviors.
- Failure of Classical Laws: As with the first-order derivative, the second-order fuzzy conformable derivative does not necessarily obey the classical index law or commutative law. This necessitates a careful approach when applying these derivatives in mathematical models.

To address these limitations, the concept of higher-order strongly generalized Ψ-differentiability can be introduced. This generalization extends the strongly generalized differentiability framework to higher orders, allowing for the differentiation of fuzzy-valued functions that may not meet the criteria for standard second-order differentiation.

Definition 4.5: Higher-Order Strongly Generalized Ψ-Differentiability

A fuzzy-valued function Φ is said to be higher-order strongly generalized Ψ-differentiable if it satisfies a set of conditions that extend beyond the standard definition of second-order differentiation. These conditions allow for the differentiation of functions that do not meet the standard criteria, thereby expanding the applicability of fuzzy conformable calculus to more complex systems.

V. FUZZY CONFORMABLE DERIVATIVE OF ORDER PΨ

The concept of fuzzy conformable derivatives can be generalized to arbitrary orders, denoted as pΨ, where p is a positive integer. This extension enables the modeling of higher-order dynamics in fuzzy-valued functions, making the framework versatile for a wide range of applications in systems characterized by complex and uncertain behaviors.

5.1 Definition of Fuzzy Conformable Derivative of Order pΨ

The fuzzy conformable derivative of order pΨ extends the ideas of first-order and second-order derivatives to any positive integer order, providing a comprehensive approach for analyzing higher-order effects in fuzzy systems.

Definition 5.1: Strongly Generalized Conformable Derivative of Order pΨ

A fuzzy-valued function Φ is said to be strongly generalized differentiable of order pΨ at a point v_0 ∈(a,b) if there exists a fuzzy number $\Phi^{p\Psi}$ (v_0) such that:

1. For any $\theta>0$, the H-differences $\Phi^{(p-1)\Psi}(v_0+\theta v^{1-\Psi}) - \Phi^{(p-1)\Psi}(v_0)$ and $\Phi^{(p-1)\Psi}(v_0) - \Phi^{(p-1)\Psi}(v_0+\theta v^{1-\Psi})$exist, and the limit

$$\lim_{\theta\to 0}\frac{\Phi^{(p-1)\Psi}(v_0+\theta v^{1-\Psi}) - \Phi^{(p-1)\Psi}(v_0)}{\theta} = \lim_{\theta\to 0}\frac{\Phi^{(p-1)\Psi}(v_0) - \Phi^{(p-1)\Psi}(v_0+\theta v^{1-\Psi})}{\theta}$$

is equal to $\Phi^{p\Psi}(v_0)$.

2. If $\Phi^{(p-1)\Psi}$ (v_0)is differentiable, the pΨ-order derivative can be expressed as:

$$\lim_{\theta\to 0}\frac{\sum_{k=0}^{p}(-1)^k{}_k^p\Phi(v_0+(p+k)\theta\, v^{1-\Psi})}{\theta^p} = \Phi^{p\Psi}(v_0)$$

This definition generalizes the fuzzy conformable derivative to any integer order, allowing for the analysis of higher-order behaviors in systems where the complexity of dynamics requires such an extension.

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5.2 Properties of the Fuzzy Conformable Derivative of Order pΨ

The fuzzy conformable derivative of order pΨ retains key properties from lower-order derivatives while extending them to accommodate the dynamics of higher-order systems.

Property 5.2: Recursive Nature of Higher-Order Derivatives

The fuzzy conformable derivative of order pΨ can be understood as a recursive application of the first-order derivative. Specifically, the pΨ-order derivative can be computed iteratively using the (p-1)Ψ-order derivative: $\Phi^{p\Psi}$ (v) = $v^{p(1-\Psi)}\Phi^{p}$ (v)

where Φ^{p} (v) denotes the p-th classical derivative of Φ with respect to v.

This recursive nature allows for a systematic extension of differentiation to higher orders, facilitating the modeling of complex, multi-order dynamics in fuzzy systems.

Taylor Series Expansion for Higher-Order Derivatives

The Taylor series can be further extended to incorporate higher-order fuzzy conformable derivatives. For an infinitely differentiable fuzzy-valued function Φ(v), the Taylor series expansion considering derivatives up to order pΨ is given by:

$$\Phi(\mathrm{v}) = \sum_{p=0}^{\infty} \frac{v^{p-p\Psi}(v-u)^{p}\Phi^{p\Psi}}{p!}(u)$$

This expansion is particularly useful in approximating fuzzy-valued functions when analyzing systems where higher-order effects are significant.

5.3 Examples and Applications

To demonstrate the utility of the fuzzy conformable derivative of order pΨ, consider the following example.

Example 5.3: Higher-Order Derivative of a Fuzzy-Valued Function

Let Φ (v) = η ×ψ(v), where η is a fuzzy number, and ψ(v) is a real-valued function that is differentiable of order pΨ. The fuzzy conformable derivative of order pΨ for Φ (v) is given by:

$\Phi^{p\Psi\,(\mathrm{v})} = \eta \times \psi^{p\Psi}(\mathrm{v})$

This example shows how the fuzzy conformable derivative of order pΨ can be applied to a product of a fuzzy number and a real-valued function, enabling the analysis of higher-order dynamics.

5.4 Limitations and Extensions

The fuzzy conformable derivative of order pΨ, like its lower-order counterparts, has certain limitations that must be considered. The complexity of higher-order derivatives can lead to challenges in computation and interpretation, particularly in systems with intricate dynamics.

Remark 5.4: Limitations and Challenges of Higher-Order Fuzzy Conformable Derivatives

- Complexity: As the order of differentiation increases, the complexity of the derivative also increases, which can make the resulting expressions difficult to interpret and apply in practice.

- Computational Challenges: Calculating higher-order fuzzy conformable derivatives may involve complex recursive processes, leading to significant computational overhead, especially in systems with high-dimensional fuzzy variables.

- Failure of Classical Properties: Similar to the first- and second-order derivatives, the fuzzy conformable derivative of order pΨ may not adhere to classical properties like the index law, commutative law, or associativity, necessitating careful analysis in applications.

Extensions

To mitigate these challenges, further generalizations and computational techniques can be developed to enhance the applicability of higher-order fuzzy conformable derivatives. These might include:

- **Approximation Methods:** Techniques for approximating higher-order derivatives to simplify computations.

- Hybrid Models: Combining fuzzy conformable calculus with other mathematical frameworks, such as neural networks or machine learning algorithms, to handle the complexity of higher-order dynamics.

Definition 5.5: Higher-Order Strongly Generalized Ψ-Differentiability

A fuzzy-valued function Φ is higher-order strongly generalized Ψ-differentiable if it satisfies conditions that allow for the differentiation beyond the standard pΨ-order framework. These conditions extend the applicability of fuzzy conformable calculus to more complex systems, where standard differentiation methods may fall short.

VI. CONCLUSION

This paper has introduced and developed the concept of fuzzy conformable derivatives, expanding the existing framework of fuzzy calculus and conformable calculus to accommodate the complexities of systems characterized by both uncertainty and fractional-order dynamics. By defining the fuzzy conformable derivative of order Ψ, 2Ψ, and extending it further to any arbitrary order pΨ, we have provided a comprehensive mathematical tool that can effectively model and analyze real-world phenomena where traditional calculus methods fall short. The fuzzy conformable derivative of order Ψ integrates the handling of vagueness inherent in fuzzy systems with the flexibility of fractional-order derivatives, offering a new approach to solving fuzzy differential equations. Extending this concept to higher orders, such as 2Ψ and pΨ, allows for the modeling of more complex systems that exhibit multi-order

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dynamics, which are common in areas like control systems, economics, engineering, and other fields dealing with imprecise data and long-memory processes.

Key contributions of this work include:

- The formal definition of fuzzy conformable derivatives for various orders, providing a unified approach to differentiate fuzzy-valued functions.
- The exploration of properties and the development of operational rules that extend classical calculus to fuzzy systems, facilitating the application of these derivatives in practical scenarios.
- Demonstrations of how these derivatives can be used to approximate functions through Taylor series expansions, and their application in solving fuzzy differential equations.

However, the study also highlights certain limitations, such as the potential unboundedness of solutions and the non-adherence to some classical mathematical laws, especially in higher-order derivatives. These challenges suggest avenues for future research, including the development of approximation methods and hybrid models that combine fuzzy conformable calculus with other computational techniques to overcome these issues. The fuzzy conformable derivative presents a powerful extension to traditional calculus, enabling more accurate and flexible modeling of uncertain and complex systems. This framework opens up new possibilities for research and applications in various scientific and engineering domains, where traditional approaches may not be sufficient to capture the full complexity of the systems under study. Future work can focus on refining these methods, exploring their computational aspects, and applying them to a broader range of real-world problems.

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