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Methods Used for Solving Linear Equation Systems

Ghulam Omer ABDULLAH
Assistant Professor, Department of Mathematics, Kabul University, AFGHANISTAN.

Correspondence Author: Ghulam Omer ABDULLAH

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ABSTRACT

The two variable equations of $a_1x + a_2y = b$ whose degree of variables is n = 1 are called linear equations. If the number of variables exceeds two (n), such a linear equation is called a n^{th} variable linear equation. If the number of equations reaches more than one (m), then existed a system of m^{th} equations, which is called a system of linear equations ($m \times n$) and shows the number of system equations and the number of variables in the system.

The system of linear equations forms the basis of linear algebra, which helps in solving and analyzing important issues in the natural sciences, especially mathematics. It is also used in solving mathematical problems with the help of computer mathematical programs. For example, it is used in the study and analysis of linear transformation as well as in solving optimization problems.

Keywords- Linear Equations, Gauss Elimination, Gauss Jordan, Cramer's Roll and LU decomposition or factorization methods.

I. INTRODUCTION

The concept of a system of linear equations was introduced first in Europe in 1637 by the famous French mathematician René Descartes, in the words of Descartes. Descartes 'appropriation was later incorporated into mathematics as Descartes' geometry. Today, with the help of this system, the routes of air transit lines at airports are determined and shown, and their intersections are calculated by solving a system of linear equations.

Suppose we have m equations and n unknowns then they form a system of the following:

$$\left(\begin{array}{l} a_{11}x_1 + a_{12}x_2 +, \dots \dots \dots, +a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 +, \dots \dots \dots, +a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 +, \dots \dots \dots, +a_{mn}x_n = b_m \end{array}\right. \quad \dots \dots \dots (1)$$

Explanation: In the above system $a_{ij}(i = 1,2,3, \dots \dots \dots, m \ , \ j = 1,2,3, \dots \dots \dots n)$ are the coefficients of the equation which are i equation number and j unknown number b_i represents constant numbers of equations and x_i represents unknown numbers in the system. If $b_i = 0$ in (1) system, such a system is called a homogeneous system. And we can show the linear system as matrixes.

II. METHODOLOGY

Since the collection of materials and information in this article is in the form of a library; therefore, the study has used a qualitative method. The researcher has compared four different methods of solving linear equation system and decided to choose the suitable method among them.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{mn} \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} : b_1 \\ a_{21} & a_{22} & \cdots a_{2n} : b_2 \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} : b_n \end{bmatrix}$$

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III. THE DIFFERENT SOLVING METHODS OF LINEAR SYSTEMS

An important issue to consider in the solving of linear systems is the consistent and inconsistent situation of linear systems. It means if a system has a solution, it is called a consistent system, and if it has no solution, it is called an inconsistent system. Of course, generally in the beginning it cannot be said that the system is consistent or inconsistent. But in the particular case, we can define the consistent and inconsistent case of Homogeneous system through the matrix rank by The Rouché –Capelli theorem.

The following four methods are used for the solution of linear systems:

1. Gauss Elimination method
2. Gauss Jordan Method
3. Cramer's Roll
4. LU Decomposition.

Before describing these methods, it is necessary to consider the following primary changes in the systems:

- Changing the location of two equations in a system
- Multiply the nonzero real number on both sides of an equation
- Adding one equation to another equation

And remove the equations from the system that take the form of compatibility.

3.1 Gauss Elimination method

Suppose the following a linear system is given;

$$\begin{cases} a_{11}x_1 + a_{12}x_2 +, \dots \dots \dots, + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 +, \dots \dots \dots, + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 +, \dots \dots \dots, + a_{mn}x_n = b_m \end{cases} \dots \dots \dots (2)$$

It should be tried to find the coefficient $(a_{11} \neq 0)$ of x_1. If the coefficient is equal to zero, then use other coefficients and remove x_1 from the system. To do this, it should be multiplied by $\frac{a_{21}}{a_{11} \neq 0}$ the both sides of first equation and it is abstracted from the second equation.

Then the both sides of first equation are multiplied by $\frac{a_{31}}{a_{11} \neq 0}$ and subtracted from the third equation, continuing this operation until x_1 becomes zero throughout the system.

We study the above method in the following m equations and n unknown system, which in the following system all of x_1 are zero.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 +, \dots \dots \dots, a_{1n}x_n = b_1 \\ \quad + a_{22}x_2 +, \dots \dots \dots a_{2n}^1 x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \\ \quad + a_{m2}x_2 +, \dots \dots \dots, a_{mn}x_n = b_m \end{cases} \dots \dots \dots (2)$$

Note: Follow the same procedure for all other unknown.

3.2 Gauss Jordan Method

To perform this method, the linear system should be arranged to (A, B) matrix and then we perform on this matrix Row Reduction operation to take the in the $(I_n x)$ form.

For example:

$$\begin{cases} x_1 + 3x_2 + x_3 = 10 \\ x_1 - 2x_2 - x_3 = -6 \\ 2x_1 + x_2 + 2x_3 = 10 \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ -6 \\ 10 \end{bmatrix}$$

$$-R_1 + R_2 \to R_2 \quad and \quad -2R_1 + R_3 \to R_3$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 10 \\ -16 \\ -10 \end{bmatrix}$$

$$if: R_2 \leftrightarrow R_3 \Rightarrow A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 5 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 10 \\ -10 \\ -16 \end{bmatrix}$$

$$-R_2 + R_3 \to R_3 \Rightarrow A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 10 \\ 10 \\ -6 \end{bmatrix}$$

$$if: \frac{R_2}{-5} and \frac{R_3}{-2} \Rightarrow A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 10 \\ 2 \\ 3 \end{bmatrix}$$

$$if: -3R_2 + R_1 \to R_1 \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$if: -1R_3 + R_1 \to R_1 \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow$$

$$x_1 = 1, x_2 = 2, x_3 = 3$$

3.3 Cramer's Roll

Using the **Cramer's** method to solve a n unknowns and m equations linear system. The condition for solving a linear system with this method is that; the determinant of the coefficients of linear system shouldn't be zero or (its matrix must be nonsingular).

This method is used for the solving of those linear systems where the number of unknowns equal with the number of equations of the system or $m = n$ and $\det A \neq 0$.

Suppose the following linear system is given:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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Assuming that the matrix A is the matrix of the coefficient unknowns and the matrix B is the matrix of the constants numbers so:

If $\det A \neq 0$ and the matrix A is reversible which its revers matrix is called $(A)^{-1}$,then this $(A)^{-1}$matrix multiplied on the both sides of the equation $AX = B$, so it can be written as:
$X = A^{-1}B$
Now with the using of adjoin matrix we can find the value of A^{-1} as:

$$A = \begin{bmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{bmatrix}, C = \begin{bmatrix} +\begin{bmatrix} a_{22}a_{23} \\ a_{32}a_{33} \end{bmatrix} - \begin{bmatrix} a_{21}a_{23} \\ a_{31}a_{33} \end{bmatrix} + \begin{bmatrix} a_{21}a_{22} \\ a_{31}a_{32} \end{bmatrix} \\ +\begin{bmatrix} a_{12}a_{13} \\ a_{32}a_{33} \end{bmatrix} - \begin{bmatrix} a_{11}a_{13} \\ a_{31}a_{33} \end{bmatrix} + \begin{bmatrix} a_{11}a_{12} \\ a_{31}a_{32} \end{bmatrix} \\ +\begin{bmatrix} a_{12}a_{13} \\ a_{22}a_{23} \end{bmatrix} - \begin{bmatrix} a_{11}a_{13} \\ a_{21}a_{23} \end{bmatrix} + \begin{bmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_{11}c_{12}c_{13} \\ c_{21}c_{22}c_{23} \\ c_{31}c_{32}c_{33} \end{bmatrix} if\, C^T \Rightarrow adjA = \begin{bmatrix} c_{11}c_{21}c_{31} \\ c_{12}c_{22}c_{32} \\ c_{13}c_{23}c_{33} \end{bmatrix}$$

Therefore, the results of the multiplication of A and C matrixes show that they $x_1, x_2, \dots, x_n$ are as follows:

$$x_1 = \frac{1}{\det A}[b_1c_{11} + b_2c_{21} + \cdots + b_nc_{n1}]$$

$$x_2 = \frac{1}{\det A}[b_1c_{12} + b_2c_{22} + \cdots + b_nc_{n2}]$$

$$x_n = \frac{1}{\det A}[b_1c_{1n} + b_2c_{2n} + \cdots + b_nc_{nn}]$$

The last formulas are called **Cramer's** formulas, for example we have for (3 × 3) square linear system:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\Delta}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\Delta}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\Delta}$$

Comment

- If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 \dots \Delta_n = 0$, then the linear system has just one solution and that solution isn't unique but its infinite.
- If $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \dots, \Delta_3 \dots \Delta_n = 0$ is nonzero then the system has no solution.
- If $\Delta \neq 0$ and the system was homogeneous i.e. $(b_i = 0)$ then the system has only zero solution, i.e.

3.4 LU decomposition or factorization method

Suppose A is a square matrix and L and U are the multiplication factors of that matrix, i.e. $A = LU$ then L and U are called the decomposition of A matrix, If we consider the above situation in terms of linear system, then we can write:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 +, \dots, a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 +, \dots, a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_1 +, \dots, a_{mn}x_1 = b_m \end{cases} \quad \dots (1)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{mn} \end{bmatrix}$$

$$AX = B \dots (2)$$

$$LUX = B \dots (3)$$

$$if: UX = Y \Rightarrow LY = B \dots (4)$$

Computation

Therefor to solve a system by the using decomposition method, we need to take the following steps for L and U matrixes:

- Finding the first line of the U matrix $(u_{11}\ u_{12}\ u_{13})$.
- Finding the first column of the L matrix (l_{21}, l_{31}).
  - Creating the second line of the matrix$(u_{22}\ u_{23})$.
  - Finding the second column of matrix (l_{32}).
  - Find the third line of the matrix (u_{33}).

For example, if we consider the following linear system, write:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$AX = B$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} c_{11} & c_{21} & \cdots & a_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{12} & u_{13} \\ 0 & 0 & u_{13} \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} =$$

$$A = LU \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

IV. CONCLUSION

By providing this research paper, we can conclude that the importance and role of linear systems in mathematical and natural sciences. Also understand what's the role of linear system in technology and engineering? This study also identified an effective method of solving equations for linear systems.

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